

# Partial entrainment in the Kuramoto-Sakaguchi model

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## **Abstract**

Synchronization of coupled oscillators is a phenomenon observed in diverse fields, from biology (chirping crickets, flashing fireflies, pacemaker cells) and sociology (hand clapping) to physics (laser arrays, Josephson junctions). For the class of systems of limit cycle oscillators a mathematical description was proposed by Winfree under the assumption that the oscillators are nearly identical and the interaction is weak. Based on this description a more specific model was proposed by Kuramoto and Sakaguchi. Kuramoto and Sakaguchi were able to perform a mathematical analysis under the additional assumption that there are an infinite number of oscillators which are randomly drawn from a given distribution. The analysis indicates the existence of a phase transition. For small values of the coupling strength the behavior of the oscillators is incoherent, while above a critical value of the coupling strength there is a solution which exhibits partial synchronization: there is a subset of oscillators moving at the same frequency value. We investigate the Kuramoto-Sakaguchi model for a finite as well as for an infinite number of oscillators.

We formulate a sufficient condition for the existence of partial entrainment, which also guarantees stability of the entrainment behavior: small perturbations of the solution exhibiting entrainment will still exhibit the same form of entrainment.

For the finite case, we describe the partial entrainment behavior for varying coupling strength, and we observe that entrainment may disappear with increasing coupling strength. This phenomenon persists in the infinite case when the frequency distribution is no longer unimodal. For the infinite case, we provide analytical arguments to support this.

The analysis also predicts another phenomenon: while partial entrainment is usually caused by large densities in the distribution of the natural frequencies of the oscillators, it may also be induced in intervals for the natural frequencies where the oscillator density itself is not high enough to account for it. The underlying mechanism can be explained in terms of resonances of the long term average frequencies with those of existing entrained subsets. The phenomenon is validated by simulation results.

**Key Words:** Kuramoto-Sakaguchi model, partial entrainment